Chapter 8 review (cont'd)

1 Problem 1

1. Evaluate:

(*i*) $\int (\ln x)^4 \, dx$

(*ii*) $\int x^2 \sin^{-1} x \, dx$

(*iii*) $\int \sqrt{e^t - 1} \, dt$

(*i*) $\int (\ln x)^4 \, dx$

First we'll substitute $w = \ln x$ so that $e^w = x$, and $e^w \, dw = dx$. Then $\int (\ln x)^4 \, dx = \int w^4 e^w \, dw$. By the tabular method for integration by parts,

u	dv
w^4	e^w
$4w^3$	e^w
$12w^2$	e^w
$24w$	e^w
24	e^w
0	e^w

The u column is derivatives of w^4, the dv column is integrals of e^w. Then,

$$\int (\ln x)^4 \, dx = \int w^4 e^w \, dw \tag{1}$$

$$= w^4 e^w - 4w^3 e^u + 12w^2 e^u - 24we^u + 24e^u + C = e^u(w^4 - 4w^3 + 12w^2 - 24w + 24) + C \tag{2}$$

$$= x((\ln x)^4 - 4(\ln x)^3 + 12(\ln x)^2 - 24 \ln x + 24 + C \tag{3}$$

(*ii*) $\int x^2 \sin^{-1} x \, dx$

We'll use integration by parts. We set u = LIPET (first log, then inv. trig, polyn., exp., trig), so $u = \sin^{-1} x$, $dv = x^2 \, dx$ implies $du = \frac{1}{\sqrt{1-x^2}} \, dx$, and $v = x^3/3$ Then,

$$\int x^2 \sin^{-1} x \, dx = (x^3 \sin^{-1} x)/3 - \int \frac{x^3}{\sqrt{1-x^2}} \, dx \tag{4}$$

To evaluate $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$ we'll set $x = \sin u$, so $dx = \cos u\,du$: then

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3 u}{\sqrt{1-\sin^2 u}} \cos du \tag{5}$$

$$= \int \frac{\sin^3 u}{\sqrt{\cos^2 u}} \cos u\,du \tag{6}$$

$$= \int \sin^3 u\,du \tag{7}$$

$$= \int \sin^2 u \sin u\,du \tag{8}$$

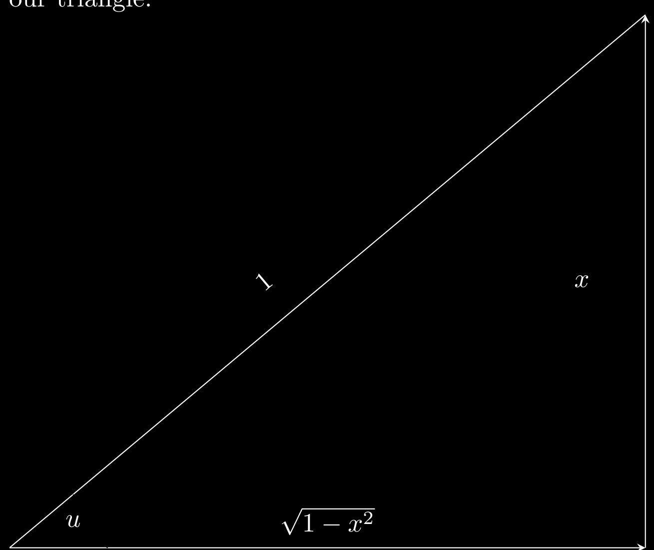
$$= \int (1-\cos^2 u) \sin u\,du \tag{9}$$

Here we set $w = \cos u$, so $dw = -\sin u\,du$. Then,

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int (1-\cos^2 u)\sin u\,du = \int (1-w^2) - dw \tag{10}$$

$$= w^3/3 - w + C \tag{11}$$

Doing our backsubs, we have $(\cos^3 u)/3 - \cos u + C = (\sqrt{1-x^2})^3/3 - \sqrt{1-x^2} + C$ since $x = \sin u$ using our triangle:



Thus $\int \frac{x^3}{\sqrt{1-x^2}}\,dx = (\sqrt{1-x^2})^3/3 - \sqrt{1-x^2} + C$, so

$$\int x^2 \sin^{-1} x\,dx = (x^3 \sin^{-1} x)/3 - \int \frac{x^3}{\sqrt{1-x^2}}\,dx \tag{12}$$

$$= (x^3 \sin^{-1} x)/3 - [(\sqrt{1-x^2})^3/3 - \sqrt{1-x^2} + C] \tag{13}$$

$$= (x^3 \sin^{-1} x)/3 - (\sqrt{1-x^2})^3/3 + \sqrt{1-x^2} + C \tag{14}$$

$$\tag{15}$$

(iii) $\int \sqrt{e^t - 1}\,dt$

Here, note $e^t - 1 \geq 0$ or else the integrand is undefined. Thus, we may let $u^2 = e^t - 1$, so $u^2 + 1 = e^t$, so that $2u\,du = e^t\,dt$, or $\frac{2u}{u^2+1}\,du = dt$ dividing by $e^t = u^2 + 1$ on both sides. Then,

$$\int \sqrt{e^t - 1}\,dt = \int \sqrt{u^2} \cdot \frac{2u}{u^2+1}\,du \tag{16}$$

$$= \int \frac{2u^2}{u^2+1}\,du \tag{17}$$

$$= \int 2 - \frac{2}{u^2+1}\,du \tag{18}$$

$$\tag{19}$$

since

$$x^2 + 1 \overline{\smash{\big)}\ 2x^2 \quad} \text{ quotient } 2; \quad -\,2x^2 - 2; \quad \text{remainder } -2$$

Note $\int 2 - \frac{2}{u^2+1}\,du = 2u - 2\tan^{-1} u + C$.

Doing our backsubs,

$$\int \sqrt{e^t - 1}\,dt = \int 2 - \frac{2}{u^2+1}\,du \tag{20}$$

$$= 2u - 2\tan^{-1} u + C \tag{21}$$

$$= 2\sqrt{e^t - 1} - 2\tan^{-1}\sqrt{e^t - 1} + C \tag{22}$$

$$\tag{23}$$

2 Problem 2

Determine whether the following integrals are improper and evaluate them.

(i) $\int_0^1 \frac{x^2}{1+x^6}\,dx$

(ii) $\int_\pi^{3\pi/2} \frac{\cos x}{1+\sin x}\,dx$

(i) $\int_0^1 \frac{x^2}{1+x^6}\,dx$

This integral is proper since the denominator is always positive on $[0, 1]$ and neither bound is infinite. We'll evaluate it:

$$\int_0^1 \frac{x^2}{1+x^6}\,dx = \int_0^1 \frac{x^2}{1+(x^3)^2}\,dx \tag{24}$$

$$= \int_{u=0^3}^{u=1^3} \frac{1}{1+u^2} \cdot \frac{1}{3}\,du \tag{25}$$

$$= [\frac{1}{3}\tan^{-1} u]_0^1 \tag{26}$$

$$= \frac{1}{3}(\pi/4) = \pi/12 \tag{27}$$

via the u-substitution $u = x^3$, so $du = 3x^2\,dx \implies \frac{1}{3}\,du = x^2\,dx$.

(*ii*) $\int_{\pi}^{3\pi/2} \frac{\cos x}{1+\sin x}\,dx$

This integral is improper since the denominator $1 + \sin x$ is zero for $x = 3\pi/2$ in our bounds. We'll thus take a limit:

$$\int_{\pi}^{3\pi/2} \frac{\cos x}{1+\sin x}\,dx = \lim_{a\to 3\pi/2^-} \int_{\pi}^{a} \frac{\cos x}{1+\sin x}\,dx \tag{28}$$

$$= \lim_{a\to 3\pi/2^-} \int_{u=1+\sin(\pi)}^{u=1+\sin a} \frac{1}{u}\,du \tag{29}$$

$$= \lim_{a\to 3\pi/2^-} [\ln|u|]_1^{1+\sin a} \tag{30}$$

$$= \lim_{a\to 3\pi/2^-} \ln|1+\sin a| - \ln 1 \tag{31}$$

$$= \lim_{a\to 3\pi/2^-} \ln|1+\sin a| = -\infty \tag{32}$$

via the u-sub $u = 1 + \sin x$, $du = \cos x\,dx$, and since

$$\lim_{x\to 0^+} \ln x = -\infty$$

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3 Problem 3

Find the smallest integer n that is guaranteed to yield an error of no more than 10^{-4} in the Trapezoidal/Simpson rules approximation of $3\int_1^2 1/x\,dx$

We know that the Trapezoidal error of f on $[a, b]$ with n subintervals is $E_n^T \le \frac{K_2}{12n^2}(b-a)^3$, with $K_2 = \max_{[a,b]} |f^{(2)}(x)|$

For us, $[a, b] = [1, 2]$, and $f(x) = 3/x$.

Then $f'(x) = -3/x^2$, $f''(x) = 6/x^3$. The maximum of $f''(x) = 6/x^3$ on [1,2] is at x=1 with $f''(1) = 6$, so $K_2 = \max_{[a,b]} |f^{(2)}(x)| = 6$.

Thus

$$E_n^T \le \frac{6}{12n^2}(2-1)^3 = \frac{1}{2n^2}$$

To guarantee this is no more than 10^{-4}, we solve the inequality: $\frac{1}{2n^2} \le 10^{-4}$ so $\sqrt{\frac{10^4}{2}} \le n$, or $\sqrt{5000} \le n$.

We know that the Simpson's error of f on $[a, b]$ with n subintervals is $E_n^S \le \frac{K_4}{180n^4}(b-a)^5$, with $K_4 = \max_{[a,b]} |f^{(4)}(x)|$, so we'll find K_4. Note $f^{(3)}(x) = -18/x^4$, $f^{(4)}(x) = 72/x^5$. This has a maximum at $x = 1$ with $f^{(4)}(1) = 72$, so $K_4 = \max_{[1,2]} |f^{(4)}(x)| = 6$.

Thus

$$E_n^S \leq \frac{72}{180n^4}(2-1)^5 = \frac{2}{5n^4}$$

To guarantee this is no more than 10^{-4}, we solve the inequality: $\frac{2}{5n^4} \leq 10^{-4}$ so $(\frac{2\cdot 10^4}{5})^{1/4} \leq n$, or $\sqrt{\sqrt{4000}} = \leq n$.

4 Problem 4

Find an upper bound for the error in approximating $\int_{-1}^{2} \frac{1}{3+x}\,dx$ by the Trapezoidal/Simpson rules for 10 subintervals.

To find an upper bound for the Trapezoidal error of $\int_{-1}^{2} \frac{1}{3+x}\,dx$, we just need to find $\frac{K_2}{12n^2}(b-a)^3$ for $n = 10$, $[a, b] = [-1, 2]$, with $K_2 = \max_{[a,b]} |f^{(2)}(x)|$, since $E_n^T \leq \frac{K_2}{12n^2}(b-a)^3$.

Well, $f'(x) = \frac{-1}{(3+x)^2}$, and $f''(x) = \frac{2}{(3+x)^3}$. The maximum of $|f''(x)|$ on $[-1, 2]$ occurs when $(3 + x)^3$ is as small as possible. This is at $x = -1$, which yields $f''(-1) = 2/2^3 = 1/4$. Thus

$$E_{10}^T \leq \frac{K_2}{12n^2}(b-a)^3 = \frac{1/4}{12(10)^2}(2-(-1))^3 = \frac{27}{4\cdot 1200} = \frac{9}{1600}$$

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To find an upper bound for the Simpson's error of $\int_{-1}^{2} \frac{1}{3+x}\,dx$, we just need to find $\frac{K_4}{180n^4}(b-a)^5$ for $n = 10$, $[a, b] = [-1, 2]$, with $K_4 = \max_{[a,b]} |f^{(4)}(x)|$, since $E_n^S \leq \frac{K_4}{180n^4}(b-a)^5$.

Well, $f'''(x) = \frac{-6}{(3+x)^4}$, and $f^{(4)}(x) = \frac{24}{(3+x)^5}$. Again the max of $|f^{(4)}(x)|$ occurs at $x = -1$ with $f^{(4)}(-1) = 24/2^5 = 3/4$. Thus

$$E_{10}^S \leq \frac{3/4}{180(10)^4}(2-(-1))^5 = \frac{3^6}{4\cdot 1800000} = \frac{9\cdot 3^4}{4\cdot 9\cdot 200,000} = \frac{81}{800,000}$$